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Problem. Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and $y \in S$, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a*c = b*c$, then $a = b$

Solution.

Proof. Let $a, b, c \in S : a * c = b * c$. Let xy be shorthand for $x * y$. We know that:

$$\begin{aligned}
&\exists e_a, e_b \in S : ae_a = a, be_b = b \\
&\exists a^{-1}, b^{-1} \in S : aa^{-1} = e_a, bb^{-1} = e_b \\
&\exists c_a^{-1}, c_b^{-1} \in S : cc_a^{-1} = e_a, cc_b^{-1} = e_b
\end{aligned}$$

This implies that:

$$\begin{aligned}
b = be_b &\Rightarrow bc = (be_b)c \Rightarrow bc = (bc)e_b \\
&\Rightarrow ac = ace_b \Rightarrow acc_a^{-1} = ae_bcc_a^{-1} \\
&\Rightarrow ae_a = ae_be_a \Rightarrow a = ae_b
\end{aligned}$$

Thus we can see that:

$$ac = bc \Rightarrow acc_b^{-1} = bcc_b^{-1} \Rightarrow ae_b = be_b \Rightarrow a = b$$

We conclude that if $a, b, c \in S, a * c = b * c \Rightarrow a = b$. $\square$