Assignment 6

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1 Chapter 6

Section 2. Problem 26
n people distributed along a road L. Let X_i denote the position of the ith person. Our goal is to compute the following probability:

$$P\{X_{(i)} > X_{(i-1)} + D, i \in \{2, 3, \ldots, n\}\}$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) \cdots f(x_n) = n! \left(\frac{1}{L}\right)^n$$

For $x_1 < x_2 < \ldots < x_n$.
Suppose $D \leq \frac{L}{(n-1)}$

$$P\{X_{(i)} > X_{(i-1)} + D, i \in \{2, 3, \ldots, n\}\} = \int \int_{x_i > x_{i-1}+D} \ldots \int f_{X_{(1)},\ldots,X_{(n)}}(x_1, x_2, \ldots, x_n)\, dx_n\, dx_{n-1} \cdots dx_1$$

$$= \frac{1}{L^n} \left(\int_0^{L-(n-1)D} \int_{x_1+D}^{L-(n-2)D} \ldots \int_{x_{n-1}+D}^{L} n!\, dx_n\, dx_{n-1} \cdots dx_1 \right)$$

Let $y_n = x_i - x_{i-1} - D$

$$= \frac{1}{L^n} \left(\int_0^{L-(n-1)D} \int_{x_1+D}^{L-(n-2)D} \ldots \int_0^{L-x_{n-1}-D} n!\, dy_n\, dx_{n-1} \cdots dx_1 \right)$$

$$= \frac{1}{L^n} \left(\int_0^{L-(n-1)D} \int_{x_1+D}^{L-(n-2)D} \ldots \int_{x_{n-2}+D}^{L-D} n!(L - x_{n-1} - D)\, dx_{n-1} \cdots dx_1 \right)$$

Now let $y_k = (L - x_k - (n-k)D)$, $\forall k \in \{1, \dots, n-1\}$, note that $dy_k = -dx_k$ so we can flip the bounds of integration:

$$= \frac{1}{L^n}\left(\int_0^{L-(n-1)D}\int_{x_1+D}^{L-(n-2)D}\cdots\int_{x_{n-2}+D}^{L-D} n!(L - x_{n-1} - D)\,dx_{n-1}\cdots dx_1\right)$$

$$= \frac{1}{L^n}\left(\int_{x_1=0}^{L-(n-1)D}\int_{y_2=L-x_1-(n-1)D}^{0}\cdots\int_{y_{n-1}=L-x_{n-2}-2D}^{0} n!y_{n-1}\,(-dy_{n-1})\,(-dy_{n-2})\cdots(-dy_2)\,dx_1\right)$$

$$= \frac{1}{L^n}\left(\int_{x_1=0}^{L-(n-1)D}\int_{y_2=0}^{L-x_1-(n-1)D}\cdots\int_{y_{n-1}=0}^{L-x_{n-2}-2D} n!y_{n-1}\,dy_{n-1}\,dy_{n-2}\cdots dy_2\,dx_1\right)$$

$$= \frac{1}{L^n}\left(\int_{x_1=0}^{L-(n-1)D}\int_{y_2=0}^{y_1}\cdots\int_{y_k=0}^{y_{k-1}}\cdots\int_{y_{n-1}=0}^{y_{n-2}} n!y_{n-1}\,dy_{n-1}\,dy_{n-2}\cdots dy_2\,dx_1\right)$$

$$= \frac{1}{L^n}\left(\int_{x_1=0}^{L-(n-1)D}\int_{y_2=0}^{y_1}\cdots\int_{y_k=0}^{y_{k-1}} \frac{n!}{(n-k)!}(y_k)^{n-k}\,dy_k\,dy_{k-1}\cdots dy_2\,dx_1\right)$$

$$= \frac{1}{L^n}\left(\int_{x_1=0}^{L-(n-1)D} n(y_1)^{n-1}\,dx_1\right) = \frac{1}{L^n}\left(-\int_{y_1=L-(n-1)D}^{0} n(y_1)^{n-1}\,dy_1\right)$$

$$= \frac{1}{L^n}\left(-\int_{y_1=L-(n-1)D}^{0} n(y_1)^{n-1}\,dy_1\right) = \frac{1}{L^n}\left(-\int_{y_1=0}^{L-(n-1)D} n(y_1)^{n-1}\,dy_1\right) = \frac{(L-(n-1)D)^n}{L^n}$$

$$= \left(1 - \frac{(n-1)D}{L}\right)^n$$

Oh my god I want to die why was that integral so long.
(*For legal reasons the above statement was not serious)

Now suppose $D > \frac{L}{(n-1)}$ Note that each person creates an open ball along the line of radius $\frac{D}{2}$ or diameter D. We would like it so that none of these balls intersect. However given that $D > \frac{L}{(n-1)}$. If we were to place our people along the line, they would create a total length of at least $D(n-1)$ on the line, which means that they necessarily must intersect given that $D(n-1) > L$. Thus $P\{X_{(i)} > X_{(i-1)} + D, i \in \{2, 3, \dots, n\}\} = 0$.
Thus:

$$P\left\{X_{(i)} > X_{(i-1)} + D, i \in \{2, 3, \dots, n\}\right\} = \begin{cases} \left(1 - \frac{(n-1)D}{L}\right)^n, & \text{if } D \le \frac{L}{(n-1)} \\ 0, & \text{otherwise} \end{cases}$$

Section 2. Problem 32

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

are the ordered values of n independent uniform $(0,1)$ random variables. Prove that for all $1 \leq k \leq n+1$,

$$P\{X_{(k)} - X_{(k-1)} > t\} = (1-t)^n$$

where $X_0 \equiv 0$, $X_{(n+1)} \equiv 1$, and $0 < t < 1$. $f_{X_{(1)},X_{(2)},\ldots,X_{(n)}} = n! \prod_{i=1}^{n} f(x) = n!$

$$\begin{aligned}
P\{X_{(k)} - X_{(k-1)} > t\} &= \int_{x_n=t}^{1} \cdots \int_{x_{k+1}=t}^{x_{k+2}} \int_{x_k=t}^{x_{k+1}} \int_{x_{k-1}=0}^{x_k-t} \cdots \int_{x_2=0}^{x_3} \int_{x_1=0}^{x_2} n!\, dx_1\, dx_2 \ldots dx_n \\
&= \int_{x_n=t}^{1} \cdots \int_{x_{k+1}=t}^{x_{k+2}} \int_{x_k=t}^{x_{k+1}} \int_{x_{k-1}=0}^{x_k-t} \frac{n!}{(k-2)!}(x_{k-1})^{k-2}\, dx_{k-1}\, dx_k \ldots dx_n \\
&= \int_{x_n=t}^{1} \cdots \int_{x_{k+1}=t}^{x_{k+2}} \int_{x_k=t}^{x_{k+1}} \frac{n!}{(k-1)!}(x_k - t)^{k-1} dx_k \ldots dx_n \\
&= \int_{x_n=t}^{1} \frac{n!}{(n-1)!}(x_k - t)^{n-1} dx_n = \frac{n!}{n!}(1-t)^n = (1-t)^n
\end{aligned}$$

2 Chapter 7

Section 1. Problem 24

1. Let $S_1, S_2, \ldots, S_n$ be the indicator functions for whether or not each pill of the n small pills remains and $B_i = 1, \ldots, m$ be the indicator functions for whether or not each of the small pills created from the ith big pill taken remains. It is clear to see that X the number of remaining small pills remaining can be written as:

$$X = \sum_{i=1}^{n} S_i + \sum_{j=1}^{m} B_j$$

Thus:

$$E[X] = \sum_{i=1}^{n} E[S_i] + \sum_{j=1}^{m} E[B_j] = n \cdot P\{S_1 = 1\} + \sum_{j=1}^{m} P\{B_j\}$$

The last equality follows from the symmetry of our variables S_n.

$$P\{S_1 = 1\} = \frac{1}{m+1}$$

The small pill must be taken at a time after all m big pills. And it is equally likely to be picked before or after any of the m big pills, so $(m+1)$ options.

$$P\{B_i = 1\} = \frac{1}{m-i+1}$$

The i small pill created by the big pill must be at a time after the remaining $m-i$ big pills, given that it is equally likely to be picked before or after any of these big pills, $(m-i)+1$ options.

$$E[X] = \frac{n}{m+1} + \sum_{j=1}^{m} \frac{1}{m-j+1} = \frac{n}{m+1} + \sum_{j=1}^{m} \frac{1}{j} = \frac{n}{m+1} + H_m$$

2. Let Y be the day on which the last large pill is chosen, note that:

$$\begin{aligned} Y &= \# \text{ of days with large pills } + \# \text{ of days with small pills} \\ &= m + (m+n-X) = 2m+n-X \end{aligned}$$

Thus:

$$E[Y] = E[2m+n-X] = 2m+n - \left(\frac{n}{m+1} + H_m\right)$$