Single-Cross 2

Frank Liu

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Problem. Consider 3-space (i.e. $\mathbb{R}^3$) partitioned into a grid of unit cubes with faces defined by the planes of all points with at least one integer coordinate. For a fixed positive real number D, a random line segment of length D (chosen uniformly in location and orientation) is placed in this cubic lattice.

What length D maximizes the probability that the endpoints of the segment lie in orthogonally adjacent unit cubes (that is, the segment crosses exactly one integer-coordinate plane), and what is this maximal probability?

Solution. WLOG, we will assume that the cube the first endpoint is in is the one in which all 3 coordinates span from 0 to 1. We can assume that $D \leq \sqrt{3}$ as increasing the length any further increases the probability of crossing through an an adjacent cube without decreasing the probability of not leaving the original cube.

For now let's assume that $D \leq 1$. Since the line is of a fixed length, we will be using spherical coordinates. Due to the rotational symmetry of the problem, we will only be integrating θ and φ over $[0 : \frac{\pi}{2}]$.
We will use the uniform density of an octant of a sphere with radius D:

$$f(\theta, \varphi) = \begin{cases} \frac{2}{\pi D^2}, & \text{if } 0 \leq \theta, \varphi \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

Let E denote the event we are in an orthogonally adjacent cube, and E_i, $i = 1, \dots, 6$ be the individual events of being in each of the 6 orthogonal cubes.

$$\begin{aligned} P(E) &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{\varphi=0}^{\frac{\pi}{2}} P(E|\Theta = \theta, \Phi = \varphi) f(\theta, \varphi) \cdot D^2 \sin(\varphi)\, d\varphi d\theta \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{\varphi=0}^{\frac{\pi}{2}} P(E|\Theta = \theta, \Phi = \varphi) \frac{2}{\pi D^2} \cdot D^2 \sin(\varphi)\, d\varphi d\theta \\ &= \frac{2}{\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{\varphi=0}^{\frac{\pi}{2}} \sum_{i=1}^{6} P(E_i|\Theta = \theta, \Phi = \varphi) \sin(\varphi)\, d\varphi d\theta \\ &= \frac{2}{\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{\varphi=0}^{\frac{\pi}{2}} 3P(E_1|\Theta = \theta, \Phi = \varphi) \sin(\varphi)\, d\varphi d\theta \end{aligned}$$

Note the last step follows from the fact that 3 of the cubes are unreachable given the angle restrictions of θ and φ. And also by the symmetry of the remaining 3.
Let (x_1, y_1, z_1) be the endpoint of the line within the cube that we are considering, then the other endpoint will be at:

$$(x_2, y_2, z_2) = (x_1 + D\sin(\varphi)\cos(\theta), y_1 + D\sin(\varphi)\sin(\theta), z_1 + D\cos(\varphi))$$

We want the second endpoint two be within an orthogonal cube (Event E_1), WLOG we are assuming it is the following region:

$$0 \le x_2 \le 1, 0 \le y_2 \le 1, 1 \le z_2 \le 2$$

Which tells us that we want the first endpoint coordinates to be in the following range:

$$\begin{aligned} -D\sin(\varphi)\cos(\theta) \le x_1 \le 1 - D\sin(\varphi)\cos(\theta) \\ -D\sin(\varphi)\sin(\theta) \le y_1 \le 1 - D\sin(\varphi)\sin(\theta) \\ 1 - D\cos(\varphi) \le z_1 \le 2 - D\cos(\varphi) \end{aligned}$$

Thus:

$$\begin{aligned} P(E) &= \frac{2}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{\frac{\pi}{2}} 3P(E_1|\Theta=\theta, \Phi=\varphi)\sin(\varphi)\,d\varphi d\theta \\ &= \frac{6}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{\frac{\pi}{2}}\int_{x_1=-D\sin(\varphi)\cos(\theta)}^{1-D\sin(\varphi)\cos(\theta)}\int_{y_1=-D\sin(\varphi)\sin(\theta)}^{1-D\sin(\varphi)\sin(\theta)}\int_{z_1=1-D\cos(\varphi)}^{2-D\cos(\varphi)} g(x_1, y_1, z_z)\sin(\varphi)\,d\varphi d\theta \end{aligned}$$

The joint density function is simply the uniform one over the cube:

$$g(x, y, z) = \begin{cases} 1, & \text{if } 0 \le x, y, z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Note $-D\sin(\varphi)\cos(\theta) <= 0$,$-D\sin(\varphi)\sin(\theta) <= 0$, and $2 - D\cos(\varphi) >= 1$, so the joint uniform density function is always 0 for that range.

$$\begin{aligned} &= \frac{6}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{\frac{\pi}{2}}\int_{x_1=0}^{1-D\sin(\varphi)\cos(\theta)}\int_{y_1=0}^{1-D\sin(\varphi)\sin(\theta)}\int_{z_1=1-D\cos(\varphi)}^{1}\sin(\varphi)\,d\varphi d\theta \\ &= \frac{6}{\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\varphi=0}^{\frac{\pi}{2}}(1 - D\sin(\varphi)\cos(\theta))(1 - D\sin(\varphi)\sin(\theta))D\cos(\varphi)\sin(\varphi)\,d\varphi d\theta \\ &= \frac{6}{\pi}\left[\frac{D(3D^2 - 16D + 6\pi)}{24}\right] = \frac{D(3D^2 - 16D + 6\pi)}{4\pi} \end{aligned}$$

Thus the probability that our segment lies in orthogonally adjacent cubes is:

$$P(E) = \frac{D(3D^2 - 16D + 6\pi)}{4\pi}$$

Now we must find the value of D that maximizes this equation. We can start by observing the critical points:

$$\frac{d}{dD}\left[\frac{D(3D^2-16D+6\pi)}{4\pi}\right]=0$$
$$\frac{d}{dD}3D^3-16D^2+6\pi D=0$$
$$9D^2-32D+6\pi=0$$

Using the quadratic formula we get:

$$D=\frac{32\pm\sqrt{32^2-4\cdot 9\cdot 6\pi}}{18}\approx 2.8103, 0.74526$$

Let us now take the second derivative and evaluate their values at the critical points.

$$\frac{d}{dD}\left[9D^2-32D+6\pi\right]=18D-32$$
$$D=2.8103, 18D-32\approx 18.585$$
$$D=0.74526, 18D-32\approx -18.585$$

Since our function is differentiable, this tells us that 0.74526 is a local maxima that decreases until hitting 2.8103 a local minima. Note that $\sqrt{3}\approx 1.73205 < 2.8103$, and the actual probability function for values of D greater than 1 will be less than the probability function when assuming $D \leq 1$ for the same values of D. Thus the probability function for all values of $D < \sqrt{3}$ (which we established as an upper bound earlier) will be less than 0.74526 the local maxima for values of $D \leq 1$.

Thus we conclude that:

$$D=\frac{32-\sqrt{32^2-4\cdot 9\cdot 6\pi}}{18}\approx 0.7452572091$$

is the length that maximizes the probability:

$$P(E)=\frac{D(3D^2-16D+6\pi)}{4\pi}\approx 0.5095346021$$